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ON THE CONTROL OF A RIGID BODY'S TRIAXIAL ORIENTATION IN THE PRESENCE OF CONSTRAINTS ON THE CONTROLS*

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The problem of controlling the triaxial orientation is investigated for a rigid body with an arbitrary mass geometry in the case when the constraints on the components of the controlling moment which is a linear combination of independent vectors are specified in implicit form. Two control problems are examined: the time-optimal (quick-acting) control problem for the reorientation of a rigid body and the design problem for a control guaranteeing the asymptotic stability of the body's motion mode being investigated. Analogous problems of the control of a rigid body's motion were investigated, in particular, in /1—3/.

1. Statement of the problem. We introduce two right-handed orthogonal coordinate systems: an inertial system $\xi\eta\zeta$ and a system xyz rigidly fixed to the body, whose axes, in the general case, do not coincide with the system of principal central axes of inertia of the rigid body. Describing the rotational motion of the body by the Euler dynamic equations

$$J\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \mathbf{M}, \quad \boldsymbol{\omega} = \{\omega_x, \omega_y, \omega_z\} \tag{1.1}$$

we define the structure of the controlling moment $\mathbf{M}$ with domain G by the relation

$$\mathbf{M} = \sum_{i=1}^{m} u_i \mathbf{M}_i; \quad G = \{\mathbf{u}\colon |u_i| \leqslant u_*, \quad i = 1, \ldots, m\}, \quad u_* = \text{const} \tag{1.2}$$

Here $\mathbf{M}_i$ are linearly-independent vectors stationary in the xyz system; the scalars u_i are the controls. If the relative position of the bases $\xi\eta\zeta$ and xyz is characterized by the quaternion $\mathbf{\Lambda} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ whose components are the Rodrigues—Hamilton parameters, then the time variation of $\mathbf{\Lambda}$ obeys the equation /4/

$$2\mathbf{\Lambda}^{\cdot} = \mathbf{\Lambda} \circ \boldsymbol{\omega} \tag{1.3}$$

When the trihedrons $\xi\eta\zeta$ and xyz coincide

$$\mathbf{\Lambda} = \mathbf{\Lambda}_* = \{\pm 1, 0, 0, 0\}$$

Problem 1. In domain G of (1.2) design a time-optimal control leading the body from the initial state

$$\mathbf{\Lambda}(t_0) = \mathbf{\Lambda}_0, \quad \boldsymbol{\omega}(t_0) = 0 \tag{1.4}$$

to the position

$$\mathbf{\Lambda}(T) = \mathbf{\Lambda}_*, \quad \boldsymbol{\omega}(T) = 0 \tag{1.5}$$

Problem 2. Having information available on the orientation parameters $\mathbf{\Lambda}$ and on the angular velocity $\boldsymbol{\omega}$, design a controlling moment $\mathbf{M}$ ensuring the asymptotic stability of the triaxial orientation mode (1.5).

2. Largest value of the controlling moment relative to a prescribed direction. In the trihedral xyz rigidly attached to the body let there be prescribed a certain $\mathbf{n}$-direction with unit vector $\mathbf{n} = \{\alpha, \beta, \gamma\}$. It will be shown below that for solving the problems posed we need to know the largest value $M_* = \|\mathbf{M}\|$ of the controlling moment relative to the prescribed $\mathbf{n}$-direction. The quantity M_* is determined from the relations

$$M_* = \max_{\mathbf{u} \in G} L(\mathbf{u}), \quad L(\mathbf{u}) = \mathbf{n}'\mathbf{M} \tag{2.1}$$

under the condition

$$\mathbf{M} \times \mathbf{n} = 0 \tag{2.2}$$

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For $m > 3$ the problem of seeking the largest value of the controlling moment relative to the prescribed $\mathbf{n}$-direction reduces to a linear programming problem. The rank of the system of constraints (2.2) equals two; therefore, at least $m-2$ of the controls $u_i (i = 1, \ldots m)$ take boundary values.

In order to study the structure of the resulting solution we restrict ourselves (without loss of generality) to the case when $m = 4$. As the vectors $\mathbf{M}_i$ $(i = 1, \ldots, 4)$ stationary in system xyz we consider the following aggregates of them:

$$\mathbf{M}_1 = \{m_x, -m_y, -m_z\}, \quad \mathbf{M}_2 = \{m_x, m_y, -m_z\}$$
$$\mathbf{M}_3 = \{-m_x, -m_y, -m_z\}, \quad \mathbf{M}_4 = \{-m_x, m_y, -m_z\}$$

The simplex method yields six groups of optimal solutions each of which is valid in a specific domain of positions of vector $\mathbf{n}$; the set

$$N = \{\mathbf{n}: \alpha^2 + \beta^2 + \gamma^2 = 1\} \tag{2.3}$$

of orientations of the unit vector $\mathbf{n}$ in the basis xyz can be represented as

$$N = \bigcup_{\nu=1}^{6} N_\nu$$

If we introduce the notation

$$d_1 = (\alpha m_y + \beta m_x) m_z \quad (123, \alpha\beta\gamma, xyz)$$
$$d_4 = (\beta m_x - \alpha m_y) m_z \quad (456, \alpha\beta\gamma, xyz)$$
$$W = 4 m_x m_y m_z u_*$$

(here and later symbols of type $(1\,2\,3, \alpha\beta\gamma, xyz)$ signify that the relations not written out are obtained by a circular permutation of the letters and indices indicated), then the subsets N_ν $(\nu = 1, \ldots, 6)$ are defined by the expressions

$$N_1 = \left\{\mathbf{n}:\ \alpha\beta \geqslant 0,\ |\gamma| \leqslant \min\left(\frac{m_z}{m_x}|\alpha|, \frac{m_z}{m_y}|\beta|\right)\right\} \quad (123, \alpha\beta\gamma, xyz)$$
$$N_4 = \left\{\mathbf{n}:\ \alpha\beta < 0,\ |\gamma| \leqslant \min\left(\frac{m_z}{m_x}|\alpha|, \frac{m_z}{m_y}|\beta|\right)\right\} \quad (456, \alpha\beta\gamma, xyz)$$

while the quantity M_* and the values u_i^0 $(i = 1, \ldots, 4)$ of the control realizing it in the subsets mentioned are found from the formulas

$$\mathbf{n} \in N_1;\quad M_* = \frac{W}{|d_1|},\quad u_1^\circ = \frac{d_6 - d_2}{|d_1|} u_*,\quad u_2^\circ = -u_3^\circ = \frac{d_1}{|d_1|} u_*,\quad u_4^\circ = -\frac{d_3 + d_5}{|d_1|} u_*$$
$$\mathbf{n} \in N_2;\quad M_* = \frac{W}{|d_1|},\quad u_1^\circ = u_3^\circ = -\frac{d_2}{|d_2|} u_*,\quad u_2^\circ = \frac{d_1 + d_6}{|d_2|} u_*,\quad u_4^\circ = \frac{d_4 - d_3}{|d_2|} u_*$$
$$\mathbf{n} \in N_3;\quad M_* = \frac{W}{|d_3|},\quad u_1^\circ = -\frac{d_2 + d_4}{|d_3|} u_*,\quad u_2^\circ = \frac{d_1 - d_5}{|d_3|} u_*,\quad u_3^\circ = u_4^\circ = -\frac{d_3}{|d_3|} u_*$$
$$\mathbf{n} \in N_4;\quad M_* = \frac{W}{|d_4|},\quad u_1^\circ = -u_4^\circ = -\frac{d_4}{|d_4|} u_*,\quad u_2^\circ = \frac{d_6 - d_5}{|d_4|} u_*,\quad u_3^\circ = -\frac{d_2 + d_3}{|d_4|} u_*$$
$$\mathbf{n} \in N_5;\quad M_* = \frac{W}{|d_5|},\quad u_1^\circ = \frac{d_6 - d_4}{|d_5|} u_*,\quad u_2^\circ = u_4^\circ = -\frac{d_5}{|d_5|} u_*,\quad u_3^\circ = -\frac{d_1 + d_3}{|d_5|} u_*$$
$$\mathbf{n} \in N_6;\quad M_* = \frac{W}{|d_6|},\quad u_1^\circ = u_2^\circ = \frac{d_3}{|d_6|} u_*,\quad u_3^\circ = -\frac{d_1 + d_2}{|d_6|} u_*,\quad u_4^\circ = \frac{d_4 - d_5}{|d_6|} u_*$$

We note that when $m = 3$ the linear form $L(\mathbf{u})$ can be given as a function of one variable and the search for M_* presents no difficulties. For example, for

$$\mathbf{M}_1 = \{m_x, 0, 0\},\ \mathbf{M}_2 = \{0, m_y, 0\},\ \mathbf{M}_3 = \{0, 0, m_z\}$$

the set (2.3) is the union of the three subsets

$$N = N_\alpha \cup N_\beta \cup N_\gamma$$
$$N_\alpha = \left\{\mathbf{n}: |\alpha| \geqslant \max\left(\frac{m_x}{m_y}|\beta|, \frac{m_x}{m_z}|\gamma|\right)\right\} \quad (\alpha\beta\gamma, xyz)$$

each of which has its own solution. Thus, for $\mathbf{n} \in N_\alpha (\alpha\beta\gamma)$

$$M_* = \frac{m_x}{|\alpha|}u_*, \quad u_1^\circ = \frac{\alpha}{|\alpha|}u_*, \quad u_2^\circ = \frac{\beta}{|\alpha|}\frac{m_x}{m_y}u_*, \quad u_3^\circ = \frac{\gamma}{|\alpha|}\frac{m_x}{m_z}u_*, \quad (\alpha\beta\gamma,\ xyz,\ 1\,2\,3)$$

3. Time-optimal control of rigid body reorientation. The problem of a time-optimal space turn of the rigid body by one turn around a fixed axis (the Euler rotation axis) has been considered, for example, in /1/. In contrast to /1/, where the constraints on the components of the controlling moment vector are known in advance, in the case we are examining these constraints are not known in advance and are determined by the orientation of the axis of final rotation in the basis rigidly attached to the rigid body. Let the orientation of the Euler rotation axis in the coordinate trihedron xyz be characterized by the direction cosines $\alpha_*, \beta_*, \gamma_*$. If the controlling moment $\mathbf{M}$ is formed such that it is colinear with the final rotation axis, then, introducing a new control by the formula

$$\mathbf{U} = J^{-1}(J\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{M}) \tag{3.1}$$

we reduce the Eq.(1.1) of rotational motion of the rigid body to the form

$$\begin{gathered} \sigma^{\cdot\cdot} = \mathbf{n}_*'\mathbf{U} \equiv U_*, \ \mathbf{n}_* = \{\alpha_*, \beta_*, \gamma_*\} \\ (\sigma^{\cdot} = \mathbf{n}_*'\boldsymbol{\omega}, \ |U_*| \leqslant U_0, \ U_0 = \text{const}) \end{gathered} \tag{3.2}$$

The problem is reduced to the construction of an algorithm for the time-optimal control leading the body from the initial state

$$\sigma(t_0) = \sigma_0, \quad \sigma^{\cdot}(t_0) = 0$$

to the final position

$$\sigma(T) = \sigma^{\cdot}(T) = 0$$

Since a one-to-one correspondence exists between the final turn vector and the Rodrigues–Hamilton parameters, by knowing the orientation of the parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ at the beginning instant of the process it is not difficult to determine the unit vector $\mathbf{n}_*$ of the Euler axis and the angle σ_0 by which the rigid body must be turned for the bases $\xi\eta\zeta$ and xyz to coincide. As a matter of fact, from the relations

$$\lambda_0(t_0) = \cos\frac{\sigma_0}{2}, \quad \lambda_1(t_0) = \alpha_*\sin\frac{\sigma_0}{2}, \quad \lambda_2(t_0) = \beta_*\sin\frac{\sigma_0}{2}, \quad \lambda_3(t_0) = \gamma_*\sin\frac{\sigma_0}{2} \tag{3.3}$$

it follows that

$$\mathbf{n}_* = \left\{\frac{\lambda_i(t_0)}{\sqrt{1-\lambda_0^2(t_0)}}\right\}, \quad i = 1, 2, 3; \quad \sigma_0 = 2\arccos\lambda_0(t_0) \tag{3.4}$$

Pontriagin's maximum principle yields the following structure of the time-optimal orientation control algorithm:

$$U_* = \varphi(\sigma, \sigma^{\cdot})U_0 \tag{3.5}$$

$$\varphi(\sigma, \sigma^{\cdot}) = \begin{cases} 1 & \text{for} \quad \sigma < \sigma_* \quad \text{and} \quad \sigma = \sigma_*, \ \sigma^{\cdot} < 0 \\ -1 & \text{for} \quad \sigma > \sigma_* \quad \text{and} \quad \sigma = \sigma_*, \ \sigma^{\cdot} > 0 \end{cases} \tag{3.6}$$

$$\sigma_* = -|\sigma^{\cdot}|\sigma^{\cdot}/(2U_0) \tag{3.7}$$

For the specified space turn of the body we find the maximum admissible value of U_0. The equality

$$\begin{gathered} \mathbf{U} = \mathbf{n}_*U_* = \varphi(\sigma, \sigma^{\cdot})U_0\mathbf{n}_* \\ \boldsymbol{\omega} = \sigma^{\cdot}n_*, \quad \sigma^{\cdot 2} = 2U_0|\sigma| \end{gathered} \tag{3.8}$$

holds for the control relative to the final turn vectors. From relation (3.1), with due regard to (3.8), we obtain the expression for the controlling moment

$$\mathbf{M} = \varphi(\sigma, \sigma^{\cdot})U_0J\mathbf{n}_* - \sigma^{\cdot 2}J\mathbf{n}_* \times \mathbf{n}_* \tag{3.9}$$

Since the largest value of the gyroscopic moment $\sigma^{\cdot 2}J\mathbf{n}_* \times \mathbf{n}_*$ is reached at the control switching instant and does not change as the function $\varphi(\sigma, \sigma^{\cdot})$ changes, we compute the vector $\mathbf{M}$ at the instant of hitting onto the switching line (3.7) when $\varphi = 1$ and $\varphi = -1$. Taking into account that when $\sigma^{\cdot}(t_0) = 0$ the maximum value of the modulus of the angular velocity $\sigma^{\cdot}$ of the rigid body's rotation around the Euler axis equals

$$|\sigma^{\cdot}|_* = \sqrt{U_0|\sigma_0|}$$

we have

$$\mathbf{M}^{\pm} = U_0(\pm J\mathbf{n}_* - |\sigma_0| J\mathbf{n}_* \times \mathbf{n}_*) \tag{3.10}$$

We note that because of the orthogonality of vectors $J\mathbf{n}_*$ and $J\mathbf{n}_* \times \mathbf{n}_*$ the vectors $\mathbf{M}^+$ and $\mathbf{M}^-$ are of the same length. However, their orientation in basis xyz, characterizable by the unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$

$$\mathbf{n}^{\pm} = [\pm J\mathbf{n}_* - |\sigma_0| J\mathbf{n}_* \times \mathbf{n}_*]/m_* \tag{3.11}$$
$$m_* = \| \pm J\mathbf{n}_* - |\sigma_0| J\mathbf{n}_* \times \mathbf{n}_* \|$$

different:

$$\mathbf{n}^- = \mathbf{n}^+ - 2J\mathbf{n}_*/m_*$$

Let the maximum values of the modulus of vector $\mathbf{M}$ relative to the $\mathbf{n}^+$- and $\mathbf{n}^-$-directions, computed in accordance with Sect.2, equal M_*^+ and M_*^-, respectively. Then

$$U_0 = \min (M_*^+/m_*, M_*^-/m_*) \tag{3.12}$$

Thus, for solving Problem 1 we first determine from expressions (3.4) and (3.10) — (3.12) the orientation in basis xyz of the Euler rotation axis, the turn angle σ_0, and the value of U_0. During the orientation by formula $\sigma = 2 \arccos \lambda_0$ we determine the current value of angle σ and we compute the required value of the controlling moment

$$\mathbf{M} = \varphi(\sigma, \sigma^{\cdot}) U_0 J\mathbf{n}_* - J\boldsymbol{\omega} \times \boldsymbol{\omega}$$

and its orientation in the attached coordinate system xyz, characterizable by the unit vector

$$\mathbf{n} = \{M_x/\|\mathbf{M}\|, M_y/\|\mathbf{M}\|, M_z/\|\mathbf{M}\|\}$$

Further, relative to the $\mathbf{n}$-direction we find the largest value M_* of the control vector and the corresponding values of $u_i^0 (i = 1, \ldots, m)$. The desired values of controls $u_i (i = 1, \ldots, m)$ are determined by the relations

$$u_i = \mu u_i^0, \quad \mu = \|\mathbf{M}\|/M_*$$

4. Asymptotic stability of the triaxial orientation mode. To solve Problem 2 we introduce the positive definite function

$$2V = \alpha V_1(\Lambda) + \boldsymbol{\omega}' J \boldsymbol{\omega}, \quad \alpha > 0 \tag{4.1}$$

vanishing at the equilibrium position (1.5) of system (1.1), (1.3). As $V_1(\Lambda)$ we can choose, for example, the function /4,5/ $V_1(\Lambda) = 2(1 - \lambda_0^2)$ or $V_1(\Lambda) = 4(1 - |\lambda_0|)$. The controlling moment $\mathbf{M}$ obtained from Liapunov theory with the aid of (4.1) can be represented as a sum of two summands

$$\mathbf{M} = \mathbf{M}_1(\Lambda) + \mathbf{M}_2(\boldsymbol{\omega})$$

We agree to choose the weighting coefficient α in (4.1) such that

$$\mathbf{M}_1(\Lambda) \in M_G = \{\mathbf{M}\colon \mathbf{u} \in G\}$$

for any orientation of the rigid body. The second component of the controlling moment is computed by the formula

$$\mathbf{M}_2(\boldsymbol{\omega}) = \varkappa_* K \boldsymbol{\omega}, \quad K < 0$$

where $\varkappa_*$ is the largest value of parameter $\varkappa$ from the range $0 < \varkappa \leqslant 1$, for which the condition

$$\mathbf{M} = \mathbf{M}_1(\Lambda) + \varkappa_* K \boldsymbol{\omega} \in M_G \tag{4.2}$$

is fulfilled. For such a controlling moment the derivative of Liapunov function (4.1) with respect to time is negative of constant sign

$$V^{\cdot} = \varkappa_* \boldsymbol{\omega}' K \boldsymbol{\omega}$$

The set $S_1 = \{\Lambda, \boldsymbol{\omega}\colon \lambda_0 \neq 0, \boldsymbol{\omega} = 0\}$ does not contain whole trajectories of the system being studied; therefore, Eq.(4.2) (with an appropriate completion of its definition at point $S_2 = \{\Lambda, \boldsymbol{\omega}: \lambda_0 = 0, \boldsymbol{\omega} = 0\}$, if this is necessary) ensures the asymptotic stability of the triaxial orientation mode /6/. We remark that when seeking $\varkappa_*$ we use the procedure, proposed in Sect.2, of computing the largest value of the controlling moment relative to the prescribed direction.

5. Examples. For the case $m = 4$ we consider the process of time-optimal space turning of a rigid body from the initial state

$$\Lambda(0) = \{0.001;\ 0.3;\ 0.6;\ 0.741619\}, \quad \boldsymbol{\omega}(0) = 0$$

to the oriented position (1.5). To the presented value of quaternion $\Lambda(0)$ corresponds the final rotation axis whose direction cosines with the axes x, y and z equal $\alpha_* = 0.3$, $\beta_* = 0.6$, $\gamma_* = 0.741620$ and the angle $\sigma_0 = 3.13959$. Typical curves of the variation of the orientation parameters, angular velocities and controls during the controled motion of the rigid body are shown in Fig.1 ($\bar{u}_i = u_i/u_*$).

If algorithm (4.2) is used for the control of the rigid body's orientation, then for $m = 3$ the typical nature of the variation of the Rodrigues–Hamilton parameters, the angular velocities and the controls during the transfer of the body from the state

$$\Lambda(0) = \{0.707;\ 0.3535;\ 0.4342;\ 0.432041\}$$
$$\omega_x(0) = \omega_z(0) = 0,\ \omega_y(0) = 0{,}5\ \mathrm{c}^{-1}$$

to the triaxial orientation mode can be seen on Fig.2.

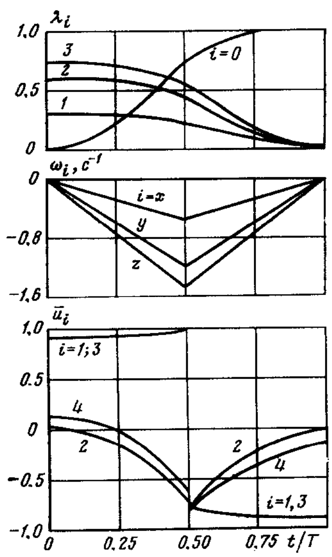

Fig.1

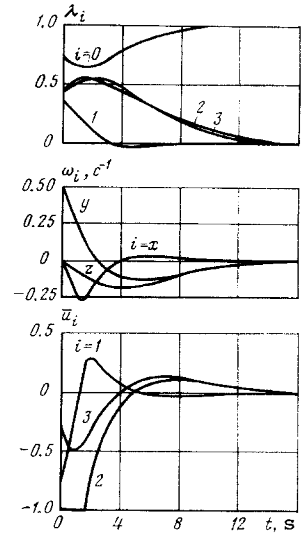

Fig.2

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